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LETTER TO THE EDITOR

A remark on the algebraic treatment of coupled oscillators

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Abstract. We determine the Levi–Malcev decompositions of some Lie subalgebras generated by a time-dependent Hamiltonian build-up by creation and annihilation operators and bilinear combinations of them, which were given incorrectly in a recent paper by Prants. The analysis shows that the calculation of the time evolution operator can be reduced to the well known case of a three-dimensional algebra plus quadratures.

In this letter we are concerned with the Lie algebraic structure of the time-dependent Hamiltonian

$$H(t)=e(t)I+\sum_{i,j=1}^{2}(\omega_{ij}(t)a_i^+a_j+b_{ij}(t)a_ia_j)+\sum_{i=1}^{2}d_i(t)a_i+\text{HC} \tag{1}$$

consisting of at most bilinear products in the bosonic creation and annihilation operators a_i^+, a_i satisfying $[a_i, a_j^+]=\delta_{ij}I$, ($I$ denoting the identity operator), $i, j=1, 2$, and scalar valued functions $\omega_{ij}(t)$, $b_{ij}(t)$, $d_i(t)$, $e(t)$, modelling time-dependent phenomena in diverse areas of physics.

The determination of the underlying algebraic structure is very useful, because the work involved in determining the time evolution operator $U(t, t_0)$ obeying the equation

$$\mathrm{i}\,\mathrm{d}U(t, t_0)/\mathrm{d}t=H(t)U(t, t_0) \qquad U(t_0, t_0)=I \tag{2}$$

can be substantially reduced by means of the Levi–Malcev decomposition (Jacobson 1962) of the Lie algebra L generated by $H(t)$ as the semidirect sum of its radical R (i.e. the maximal solvable ideal of L) and a semisimple subalgebra S (S can be further split into its simple components, but we will not need this here). This decomposition gives rise to a decoupling of (2) of the form (Wei and Norman 1963)

$$\begin{aligned}
&\mathrm{i}\frac{\mathrm{d}U_S(t, t_0)}{\mathrm{d}t}=H_S(t)U_S(t, t_0)\\
&\mathrm{i}\frac{\mathrm{d}U_R(t, t_0)}{\mathrm{d}t}=\{U_S^{-1}(t, t_0)H_R(t)U_S(t, t_0)\}U_R(t, t_0)
\end{aligned} \tag{3}$$

$H_R\in R$, $H_S\in S$. U is then given as a product $U=U_SU_R$, where U_R can be determined by quadratures.

In a recent article (Prants 1986) decompositions of some subalgebras of the Lie algebra generated by $H(t)$ were incorrectly determined.

Decompositions were given for the cases

(i) $\omega_{12}=\omega_{21}=b_{11}=b_{22}=e=0$

$$L_1:=\{a_1^+a_1+\tfrac{1}{2},\ a_2^+a_2+\tfrac{1}{2},\ a_1^+a_2^+,\ a_1a_2,\ \mathrm{n}(2)\}$$

(ii) $b_{ij}=0 \qquad i,j=1,2 \qquad e=0$

$$L_2:=\{a_1^+a_1+\tfrac{1}{2},\ a_2^+a_2+\tfrac{1}{2},\ a_1^+a_2,\ a_1a_2^+,\ \mathrm{n}(2)\}$$

with $\mathrm{n}(2):=\{a_1, a_1^+, a_2, a_2^+, I\}$.

In the cited paper the L_i, $i=1,2$ were decomposed as

$$L_1=\mathrm{u}(1,1)\oplus\mathrm{n}(2)$$

$$L_2=\mathrm{u}(2)\oplus\mathrm{n}(2)$$

with

$$\mathrm{u}(1,1)=\{a_1^+a_1+\tfrac{1}{2},\ a_2^+a_2+\tfrac{1}{2},\ a_1^+a_2^+,\ a_1a_2\}$$

and

$$\mathrm{u}(2)=\{a_1^+a_1+\tfrac{1}{2},\ a_2^+a_2+\tfrac{1}{2},\ a_1^+a_2,\ a_1a_2^+\}.$$

But these are no Levi–Malcev decompositions, as can be readily seen by calculating the Killing forms $K(x_i,x_j):=\mathrm{Tr}(\mathrm{ad}(x_i)\,\mathrm{ad}(x_j))$, $i,j=1,\ldots,4$, for both algebras, which have vanishing determinant. Here x_i, x_j denote the elements of $\mathrm{u}(1,1)$ and $\mathrm{u}(2)$, respectively. Therefore the algebras are not semisimple and the radicals of L_i, $i=1,2$, have not been found.

The problem involved with the determination of the radical is that sometimes it is not simple to read it off from the multiplication table of a Lie algebra L, because in the chosen basis of L, although convenient for physical applications, the basis of the radical might be disguised in the structure constants not detectable by pure inspection. In the following we will derive a system of linear algebraic equations by using a constructive definition of the radical of a Lie algebra.

Over $\mathbb{R}$ or $\mathbb{C}$ an equivalent definition of R to the one given above is (Jacobson 1962)

$$R:=\{x\in L \mid K(y,x)=0,\ \text{for all } y\in[L,L]\}.$$

If we fix a basis of $[L,L]$, i.e. $[L,L]=\{y_1,\ldots,y_M\}$, $M=\dim[L,L]$ one obtains M equations determining R.

$$\begin{aligned} &K(y_1,x)=0\\ &\ \ \vdots\\ &K(y_M,x)=0. \end{aligned} \tag{4}$$

We express an arbitrary element $x\in L$ as a linear combination of the chosen basis, i.e. $x=\sum_{i=1}^{N}p_ix_i$, $p_i\in\mathbb{R}$ or $\mathbb{C}$, $N=\dim L$. Inserting this into (4) yields

$$\begin{aligned} &K(y_1,x_1)p_1+\ldots+K(y_1,x_N)p_N=0\\ &\ \ \vdots\\ &K(y_M,x_1)p_1+\ldots+K(y_M,x_N)p_N=0. \end{aligned} \tag{5}$$

(5) gives us the coefficients, which determine what kind of linear combinations of the basis elements of L span R. The calculations involved decompose L into

$$L=R\oplus L/R \qquad L/R \text{ semisimple.}$$

If L/R is closed under commutation the decomposition is complete, otherwise one has to modify the basis of L/R to obtain closure (Rand *et al* 1986).

For the algebras in question we arrive at the following decompositions:

$$L_1 = \{\mathrm{n}(2), a_1^+a_1 - a_2^+a_2\} \oplus \{a_1^+a_2^+, a_1a_2, a_1^+a_1 + a_2^+a_2 + I\}$$

$$L_2 = \{\mathrm{n}(2), a_1^+a_1 + a_2^+a_2\} \oplus \{a_1^+a_1 - a_2^+a_2, a_1a_2^+, a_1^+a_2\}$$

where the first bracket in each sum gives the radical. The semisimple subalgebras are three-dimensional, i.e. both subalgebras are already simple.

The solutions of (3) can be read off from Wei and Norman (1963) in the case of the simple parts, which shows another advantage of using the structure theory of Lie algebras. Since the simple algebras are classified, the time evolution operators have to be determined only once. A detailed study of Lie algebras generated by (1) will be published elsewhere.

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